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Inverse Problem for Periodic "Weighted" Operators

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Define the periodic weighted operator $Ty=-\rho^{-2}(\rho^2 y')'$ in $L^2(\mathbb{R}, \rho(x)^2\,dx)$. Suppose a function $\rho\in W_1^2(\mathbb{R}/\mathbb{Z})$ is 1-periodic real positive, $\rho(0)=1$, and let $q=\rho'/\rho\in L^2(0,1)$. The spectrum of T consists of intervals $\sigma_n=[\lambda_{n-1}^+, \lambda_n^-]$ separated by gaps $\gamma_n=(\lambda_n^-, \lambda_n^+)$, $n\geqslant 1$, with the lengths $|\gamma_n|$ and $\lambda_0^+=0$. Let m_n^2, $n\geqslant 1$, be the Dirichlet eigenvalue of the equation $-y''-2qy'=z^2 y$, $y(0)=y(1)=0$ where $m_n>0$. Introduce the Lyapunov function $\Delta(z,q)$ for T and note that $\Delta_z(z_n,q)=0$ for some $z_n\in[\sqrt{\lambda_n^-}, \sqrt{\lambda_n^+}]$. Let $\varphi(x,z,q)$ be the solution of the equation $-\varphi''-2q\varphi'=z^2\varphi$, $z\in\mathbb{C}$, satisfying $\varphi(0,z,q)=0$, $\varphi'(0,z,q)=1$. Introduce the vector $h_n=(h_{cn}, h_{sn})\in\mathbb{R}^2$, with components $h_{cn}=-\log[(-1)^n\varphi'(1,m_n,q)]$, $h_{sn}=|\,|h_n|^2-(h_{cn})^2|^{1/2}\operatorname{sign}(z_n-m_n)$, where $|h_n|$ is defined by the equation $\cosh|h_n|=(-1)^n\Delta(z_n,q)\geqslant 1$ and coincides with the euclidien norm of the vector h_n. Using nonlinear functional analysis in Hilbert space, we prove that the mapping $h\colon q\to h(q)=\{h_n\}_1^\infty$ is a real analytic isomorphism. © 2000 Academic Press

1. INTRODUCTION AND MAIN RESULTS

Consider the periodic weighted operator

$$Tf=-\rho^{-2}(\rho^2 f')'=-f''-2qf', \qquad q\equiv\frac{\rho'}{\rho}\in L^2(0,1), \tag{1.1}$$

in the Hilbert space $L^2(\mathbb{R},\rho(x)^2\,dx)$ with the norm $\|f\|_{\rho^2}^2=\int_0^1\rho^2(x)\,|f(x)|^2\,dx$, where $\rho\in W_1^2(\mathbb{R}/\mathbb{Z})$ is a 1-periodic real positive function, i.e., $\rho(x)>\rho_0$, $x\in[0,1]$ for some $\rho_0>0$. Without loss of generality, assume $\rho(0)=1$. It is well known (see [L, Kr]) that the spectrum of T is absolutely continuous and consists of intervals $\sigma_n=[\lambda_{n-1}^+,\lambda_n^-]$, where $\lambda_{n-1}^+\leqslant\lambda_n^-\leqslant\lambda_n^+$, $n\geqslant 1$. These intervals are separated by the gaps $\gamma_n=(\lambda_n^-,\lambda_n^+)$, with the length $|\gamma_n|\geqslant 0$. If a gap γ_n is degenerate, i.e., $|\gamma_n|=0$, then the corresponding

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segments σ_n, σ_{n+1} merge. In analogy to the notation $O(1/n)$ we use the notation $\ell^d(n)$, $d\geqslant 1$, for an arbitrary sequence of numbers, which is an element of ℓ^d (see [PT]). For instance,

$$a_n=b_n+\ell^d(n) \text{ is equivalent to } a_n=b_n+c_n, \qquad \sum |c_n|^d<\infty.$$

Let us recall the author's result [K1]

$$\begin{aligned} z_n^{\pm} &\equiv \sqrt{\lambda_n^{\pm}} \\ &=\pi n \pm \left|\int_0^1 e^{i2\pi n x} q(x)\,dx\right| + \ell^d(n), \qquad d>1, \quad \text{as } n\to\infty, \end{aligned} \tag{1.2}$$

and these asymptotics give the estimates of gap lengths. Roughly speaking these asymptotics give the main difference between the spectral properties of the operator T and the Hill operator: the gap length $|\gamma_n|\to\infty$ as $n\to\infty$ for T while the gap length of the Hill operator tends to zero at large n. Introduce the fundamental solutions $\varphi(x, z, q)$, $\vartheta(x, z, q)$ of the equation

$$-y''-2qy'=z^2y, \qquad z\in\mathbb{C}, \tag{1.3}$$

with the conditions $\varphi(0, z, q)=\vartheta'(0, z, q)=0$, $\varphi'(0, z, q)=\vartheta(0, z, q)=1$, here and below $(')=\partial/\partial x$, $(\dot{})=\partial/\partial z$, $\partial=\partial/\partial q$. The Lyapunov function is defined by

$$\Delta(z, q)=\tfrac{1}{2}(\varphi'(1, z, q)+\vartheta(1, z, q)). \tag{1.4}$$

The sequence $\lambda_0^+<\lambda_1^-\leqslant\lambda_1^+<\cdots$ is the spectrum of Eq. (1.3) with the periodic boundary conditions of period 2, i.e., $f(x+2)=f(x)$, $x\in\mathbb{R}$. Here equality means that $\lambda_n^-=\lambda_n^+$ is a double eigenvalue. We note that $\Delta(z_n^{\pm}, q)=(-1)^n$, $n\geqslant 1$. The lowest eigenvalue $\lambda_0^+=0$ is simple, $\Delta(0, q)=1$, and the corresponding eigenfunction has period 1. The eigenfunctions corresponding to $\lambda_n^{\pm}$ have period 1 when n is even and they are anti-periodic, $f(x+1)=-f(x)$, $x\in\mathbb{R}$, when n is odd. For each $n\geqslant 1$ there exists a unique point $z_n\in[z_n^-, z_n^+]$ such that $\dot{\Delta}(z_n, q)=0$.

Let $\mu_n(q)$, $n\geqslant 1$, be the Dirichlet spectrum of q, that is, the spectrum of (1.3) with boundary condition $y(0)=y(1)=0$, and let $m_n(q)=\sqrt{\mu_n(q)}>0$. It is well known that $m_n(q)\in[z_n^-, z_n^+]$, $n\geqslant 1$.

There are various methods of solving inverse problems. We briefly describe a "direct approach" from [KK1], based on a theorem from non-linear functional analysis. We recall definitions. Suppose that H, H_1 are real separable Hilbert spaces with norms $\|\cdot\|$, $\|\cdot\|_1$. The derivative of a map $f\colon H\to H_1$ at a point $y\in H$ is a bounded linear map from H into H_1, which we denote by $d_y f$. A map $f\colon H\to H_1$ is compact on H if it maps weakly convergent sequences in H into strongly convergent sequences in H_1. A map $f\colon H \to H_1$ is a real analytic isomorphism between H and H_1 if f is one-to-one and onto and both f and f^{-1} are real analytic maps of the Hilbert space. Let H_C be the complexification of the real Hilbert space H with the norm $\|\cdot\|_C$.

We formulate a new version of a "basic theorem" of the direct method (see [KK1, KK2]).

THEOREM A. *Let H, H_1 be real separable Hilbert spaces with norms $\|\cdot\|$, $\|\cdot\|_1$. Suppose that a map $f\colon H \to H_1$ satisfies the following conditions*:

(i) *f is real analytic,*

(ii) *the operator $d_q f$ has an inverse for all $q \in H$,*

(iii) *there is a nondecreasing function $g\colon [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that $\|q\| \leqslant g(\|f(q)\|_1)$ for all $q \in H$,*

(iv) *there exists a basis $\{e_n\}_1^\infty$ of H_1 such that each map $(f(\cdot), e_n)_1\colon H \to \mathbb{R}$, $n \geqslant 1$, is compact.*

(v) *for each $\zeta > 0$ the set $\{q\colon \sum n^2 (f(q), e_n)_1^2 < \zeta\} \subset H$ is compact.*

Then f is a real analytic isomorphism between H, H_1.

Remark. For the Hill operator instead of Condition (v) we have the following condition (see [KK1, KK2])

(v′) *there exists a linear isomorphism $J\colon H \to H_1$ such that $(f(q) - Jq, e_n) = O(1/n)$ as $n \to \infty$ uniformly on bounded subsets of H.*

Introduce the real Hilbert spaces $H = \{q\colon q \in L^2(0, 1), \int_0^1 d(x)\, dx = 0\}$ and

$$\ell^r = \left\{ f = \{f_n\}_1^\infty,\ f_n \in \mathbb{R},\ \|f\|_r^r = \sum_{n \geqslant 1} |f_n|^r < \infty \right\}, \qquad r \geqslant 1,$$

and define the Fourier transformation $\Phi\colon H_C \to \ell_C^2 \oplus \ell_C^2$ such that $(\Phi q)_n = \sqrt{2}\, q_n$, $n \geqslant 1$, where $q_n = (q_{cn}, q_{sn})$, and the coefficients q_{cn}, q_{sn} have the form

$$q_{cn} = \int_0^1 q(x) \cos 2\pi n x\, dx,$$

$$q_{sn} = \int_0^1 q(x) \sin 2\pi n x\, dx, \qquad n \geqslant 1, \quad q \in H_C.$$

Now we construct the mapping h: $q \to h(q) = \{h_n\}_1^\infty$ from H into $\ell^2 \oplus \ell^2$ by the following rule: $h_n = (h_{cn}, h_{sn}) \in \mathbb{R}^2$, where the function $|h_n(q)|^2$, $q \in H$ is defined by the equation

$$\cosh |h_n(q)| = (-1)^n \Delta(z_n(q), q), \qquad q \in H, \tag{1.5}$$

and the components have the form

$$\begin{aligned} h_{cn}(q) &= -\log[(-1)^n \varphi'(1, m_n(q), q)], \\ h_{sn}(q) &= |\,|h_n(q)|^2 - h_{cn}^2(q)|^{1/2} \operatorname{sign}(z_n(q) - m_n(q)). \end{aligned} \tag{1.6}$$

Using the identity $\varphi'\vartheta - \vartheta'\varphi = 1$ and the relation $\varphi'(1, m_n(q), q) = (-1)^n e^{-h_{cn}(q)}$ we deduce that $\vartheta(1, m_n(q), q) = (-1)^n e^{h_{cn}(q)}$ and then

$$(-1)^n \Delta(m_n(q), q) = \cosh h_{cn}(q), \qquad n \geqslant 1. \tag{1.7}$$

Note that $h_n^2 - h_{cn}^2 \geqslant 0$, since $(-1)^n \Delta(z, q)$ has the maximum at z_n on the segment $[z_n^-, z_n^+]$. Define the ball $B_C(p, t) = \{q: \|q - p\|_C < t\} \subset H_C$, $p \in H_C$, and introduce the constant ε_q by the formula $\varepsilon_q = 8^{-2} e^{-4\|q\|}$. Our main goal is to solve the inverse problem for the mapping $h(q)$ by a direct method. We formulate our main result.

THEOREM 1.1. *The mapping h: $H \to \ell^2 \oplus \ell^2$ is a real analytic isomorphism. Each function $h_n(\cdot)$, $n \geqslant 1$, is compact and real analytic on H. Moreover, for any fixed $d > 1$ the following estimates are fulfilled;*

$$\|q\| \leqslant \|h\| \leqslant \|q\| (1 + \|q\|), \qquad q \in H, \tag{1.8}$$

and

$$h_n(q) = q_n + \ell^d(n), \qquad n \to \infty, \tag{1.9}$$

$$(d_q h_n(q))(x) = (\cos 2\pi n x, \sin 2\pi n x) + \ell^2(n), \qquad n \to \infty, \tag{1.10}$$

uniformly on $[0, 1] \times B_C(p, \varepsilon_p)$ for each fixed $p \in H$.

Remark. (i) Recall that in analogy to the notation $O(1/n)$ we use the notation $\ell^d(n)$, $d \geqslant 1$. For example, (1.10) means that there exists a sequence $\beta = \{\beta_n(q)\} \in \ell^2$ such that $\|\beta\|_d \leqslant 1$ and

$$(d_q h_n(q))(x) = (\cos 2\pi n x, \sin 2\pi n x) + \beta_n(q)\, O(1), \qquad n \to \infty,$$

uniformly on $[0, 1] \times B_C(p, \varepsilon_p)$ for each fixed $p \in H$. (ii) The estimates (1.8) were proved in [K1, K3]. (iii) In order to prove this theorem we use the results from [K2], where the author solved the same problem for the Hill operator (after [MO1, MO2]) by the direct method. (iv) Roughly speaking in Theorem 1.1 the result of [MO1, MO2] is extended to singular potentials (distributions), see formula (1.16). This generalization requires essentially different proofs, since the technique that was used for studying the Hill operator could not be directly generalized to the weighted operator. This induced us to reconsider the original version of the proof (Theorem A), this led us evidently to discover simplifications, and also improvements in the estimates. The proof presented here depends on the investigation of the fundamental solutions and some integral equation (see (2.2)) at large $|z|$, which seem to us also have independent interest. In the proof of asymptotics estimates there exists a big difference between the Hill operator and the weighted operator. Roughly speaking in the integral equation for the fundamental solution of the Hill operator there exists the small parameter $1/z$! But such small parameter $1/z$ is absent for the weighted operator (see (2.2)). Some asymptotics estimates was found in [K1], but the main part of needed ones is proved here.

Introduce a quasimomentum domain $K=\mathbb{C}\setminus\cup\Gamma_n$, where $\Gamma_n=(\pi n+i\,|h_n(q)|,\pi n-i\,|h_n(q)|)=-\Gamma_{-n}$ is an excised slit with height $|h_n(q)|\geqslant 0$, $n\geqslant 1$, and $h_0=0$. Below we need the spectral domain $\mathscr{Z}=\mathbb{C}\setminus\cup g_n$ where $g_n=(z_n^-, z_n^+)=-g_{-n}$, $n\geqslant 1$, is a horizontal slit with length $|g_n|\geqslant 0$ and we recall $z_n^\pm=\sqrt{\lambda_n^\pm}>0$.

Corollary 1.2. *For each $h\in\ell^2\oplus\ell^2$ there exist a unique function $q\in H$ and a unique conformal mapping k: $\mathscr{Z}\to K$ such that following identities and asymptotics are fulfilled*:

$$\cos k(z)=\Delta(z,q),\qquad z\in\mathscr{Z},\tag{1.11}$$

$$k(z)=z-\frac{\|q\|^2+o(1)}{2z},\qquad |\mathrm{Im}\, z|\to\infty,\tag{1.12}$$

$$k(z_n\pm i0)=\pi n\pm i\,|h_n|,\qquad n\geqslant 1,\tag{1.13}$$

$$k(m_n(q)\pm i0)=\pi n\pm ih_{cn},\qquad n\geqslant 1.\tag{1.14}$$

Proof. In Theorem 1.1 we proved that for each $h\in\ell^2\oplus\ell^2$ there exists a unique potential $q\in H$ such that the identities (1.6) are fulfilled. Moreover, in [K1] the author constructed a unique conformal mapping k: $\mathscr{Z}\to K$ with the needed properties (1.11)–(1.13). Then using (1.7) we obtain (1.14). ∎

Define the unitary transformation U: $L^2(\mathbb{R},\rho^2\,dx)\to L^2(\mathbb{R})$ by the formula $(Uf)=\rho f$. Then we introduce the operator

$$T_1=UTU^{-1}=-\left(\frac{d}{dx}+q\right)\left(\frac{d}{dx}-q\right)\tag{1.15}$$

and if $q' \in L^2(\mathbb{R}/\mathbb{Z})$, then

$$T_1 = -\frac{d^2}{dx^2} + V, \qquad V = q'(x) + q(x)^2 = \frac{\rho''(x)}{\rho(x)}, \tag{1.16}$$

and hence

$$\|V\|^2 = \|q'\|^2 + \|q^2\|^2, \qquad \int_0^2 V(x)\,dx = \|q\|^2. \tag{1.17}$$

For q we define the new functions $\tilde{q}$, $\tilde{\rho}$ and the constant N_q by the formulas

$$\begin{gathered}\tilde{\rho}(x) = \exp\int_0^x \tilde{q}(t)\,dt, \qquad \tilde{q}(x) = \sum_{-m}^{m} (q_{cn} - iq_{sn})\,e^{i2\pi nx},\\ N_q = 1 + 8\,\|q\|\,[2\pi m + \|q\|]\,e^{4\,\|q\|},\end{gathered} \tag{18}$$

where m is defined by the following condition: $\|q - \tilde{q}\| \leqslant \varepsilon$. The function $\tilde{q}$ is smooth and the unitary transformation leads to the Hill operator with the smooth periodic potential

$$\tilde{V} = \tilde{\rho}''/\tilde{\rho} = \tilde{q}' + \tilde{q}^2. \tag{1.19}$$

Then

$$\int_0^1 |\tilde{V}(x)|\,dx \leqslant \int_0^1 |\tilde{q}(x)'|\,dx + \|\tilde{q}\|^2 \leqslant 2\pi m\,\|q\| + \|q\|^2. \tag{1.20}$$

First results about periodic weighted operator T were obtained by Lyapunov [L]. He proved that the spectrum of T has the band structure. Later Krein reproved this result in a more general case including 2×2 systems. The global quasimomentum was introduced into the spectral theory of the Hill operator by Firsova [F] and by Marchenko and Ostrovski [MO1] simultaneously. The author [K1] obtained the following basic result about "the direct problem" for periodic weighted operators: (i) asymptotics of various parameters (see (1.2)), (ii) double-sided estimates, (iii) the global quasimomentum. It is these results that we use in the present paper. Note that it is not clear how to get the asymptotics of the fundamental solutions for the case $\rho \in C(\mathbb{R}/\mathbb{Z})$.

A great numbers of papers are devoted to the inverse problem for the Hill operator, Marchenko and Ostrovski [MO1, MO2] proved that the mapping h: $V \to h(V)$ is continuous isomorphism. Garnett and Trubowitz [GT] proved that this mapping is the real analytic isomorphism for the case of even potentials. Kargaev and the author [KK1] reproved the result of Garnett and Trubowitz [GT] by a direct method. Note that the proofs by the direct method are short but this approach needs some estimates (see (iii) in Theorem A and (1.8)). The author [K2] proved that $h\colon V \to h(V)$ is real analytic isomorphism by the direct method. Double-sided estimates for various parameters of the Hill operator (the norm of a periodic potential, effective masses, gap lengths, height of slits $|h_n|$ and so on) were obtained in [K4–K6] and for the Dirac operator in [KK3, K3, K5]. The precise double sided estimates for gap lengths was found in [K7]. Pöschel and Trubowitz [PT] wrote a book concerning the inverse Dirichlet problem. Coleman and McLaughlin [CM] solved the Dirichlet inverse problem for Eq. (1.3). In the present paper we need more strong estimates than in [CM], see Lemma 2.1 and Lemma 2.5.

We finish the introduction by briefly explaining how the proof by a direct approach will go, i.e., how we verify conditions (i)–(v) of Theorem A for the mapping h. Remark that the theorem from [KK1, KK2] does not work since there exists a difference between the asymptotics of h_n, $n \to \infty$: $h_n = (2\pi n)^{-1} V_n + O(n^{-2})$, for the Hill operator and $h_n = q_n + \ell^d$, $d > 1$, $n \to \infty$, for the weighted operator (see (1.9)). Asymptotics for the Hill operator is more convenient than asymptotics for the weighted operator. Note that in the proof we only use some results from [K2, PT] and the estimates (1.8) from [K1] (see also [K3]). The checking of (i) is based on the analyticity of the functions $\varphi(\cdot, z, q)$, $\vartheta(\cdot, z, q)$, $\Delta(z, q)$ of $z \in \mathbb{C}$, $q \in H_C$. In order to obtain the analytic of the mapping h we need to prove the real analyticity of two mappings $z_n(\cdot)$, $|h_n(\cdot)|^2$. The main problem is connected with the degeneracy of the gap $h_n(q)$, i.e., the case $z_n^- = z_n^+$, and then with the high energy gaps. To overcome these difficulties we show the analyticity of the mappings in the ball $B_C(q, \varepsilon_q)$ for any fixed $q \in H$ and all $n \gg 1$.

To check (ii), we prove that the Fréchet derivations of the map is a Fredholm operator of index zero with zero-dimensional kernels and, therefore, is invertible. Here we use a result of Paley–Wiener about entire functions (see Lemma 4.3) and the following fact: for any fixed $q \in H$ the vectors $\{(d_q h_{cn}), (d_q \mu_n), n \geqslant 1\}$ form a basis of H (see [CM]). We emphasize the important role of the entire function $\partial\Delta(z, q)$. The verification of (iii) is based on estimates (1.8). To check (iv), we use the compactness of the mappings $z_n^{\pm}(\cdot)\colon H \to \mathbb{R}$, $\mu_n(\cdot)\colon H \to \mathbb{R}$ and the fundamental solutions. The checking of (v) is based on estimates given in [K6].

In conclusion we relate the operator T with other spectral problems. Using a unitary map, the operator T can be transformed, even with non-smooth coefficients, by

$$x = x(t) = \int_0^t \rho(t)^{-2}\, dt, \qquad c(x) = \rho(t(x))^{-2},$$

where $t(x)$ is the inverse function $x(t)$, into the following weighted operator

$$T_c f = -c^2(x) f'' \qquad \text{in} \quad L^2(\mathbb{R}, c(x)^{-2}\, dx), \tag{1.21}$$

where $c(x)$ is the sound velocity in the periodic media. The operator T_c is also studied in the photonic crystal problem, where $\varepsilon(x) = c(x)^{-2}$ is the dielectric coefficient, periodic in x (see [N]). Remark that the equation $-\ddot{y} = z^2 b(t)^2\, y$ arises in the scattering of the few particles in an external homogeneous time-periodic magnetic field $B(t) = zb(t)(0, 0, 1) \in \mathbb{R}^3$, where z is the amplitude and $b(\cdot)$ is 1-periodic (see [K8]).

2. PROPERTIES OF FUNDAMENTAL SOLUTIONS

Define the functions

$$f_+ = \varphi' - i\frac{\vartheta'}{z}, \qquad f_- = \varphi' + i\frac{\vartheta'}{z},$$

and then

$$\varphi' = \frac{1}{2}(f_+ + f_-), \qquad \vartheta' = \frac{z}{2i}(f_- - f_+). \tag{2.1}$$

Note that for each x the function $f_\mp(x, z, q)$ is entire (see Lemma 2.1). The function $f_\pm(x, z, q)$ is the solution of the equation

$$f_\pm(x, z, q) = e^{\pm izx} - \int_0^x \cos z(x-t)\, 2q(t)\, f_\pm(t, z, q)\, dt,$$

$$x \in \mathbb{R}, \quad z \in \mathbb{C}, \quad q \in H_C, \tag{2.2}$$

which we rewrite in the short form

$$f_\pm = e_\pm - A_+ f_\pm - A_- f_\pm, \tag{2.3}$$

where

$$e_\pm = e^{\pm izx},$$

$$(A_\pm(z, q)\, f)(x) = e^{\pm izx} \int_0^x e^{\mp izt} q(t)\, f(t)\, dt, \qquad z \in \mathbb{C}, \quad f, q \in H_C.$$

The solution of the equation

$$-y'' - 2qy' - z^2 y = f$$

has the form

$$y = C_1 \vartheta(x, z, q) + C_2 \varphi(x, z, q) + \int_0^x G(x, t, z, q)\, f(t)\, dt,$$
$$G(x, t, z, q) = \rho(t)^2\, (\vartheta(x, z, q)\, \varphi(t, z, q) - \vartheta(t, z, q)\, \varphi(x, z, q)). \tag{2.4}$$

Define the Sobolev space $W_2^2(0, 1)$ of functions f such that $f, f'' \in L^2(0, 1)$. We shall need the following results from [K1].

LEMMA 2.1. *For each $x \in [0, 1]$ the functions $f_\pm(x, z, q)$ are entire on $\mathbb{C} \times H_C$. Moreover, the functions $f_\pm$ are analytic as a map from $\mathbb{C} \times H_C$ into $W_{1C}^2(0, 1)$ and the following estimates are fulfilled,*

$$|f_\pm(x, z, q)| \leqslant e^{|\mathrm{Im}\, z|\, x + 2\, \|q\|_C}. \tag{2.5}$$

If the sequence q^ν converges weakly to q in H_C, as $\nu \to \infty$, then $f_\pm(x, z, q^\nu) \to f_\pm(x, z, q)$ uniformly on bounded subsets of $[0, 1] \times \mathbb{C}$. Moreover, for any fixed $d > 1$ we have

$$f_\pm(x, z, q) = \frac{e^{\pm izx}}{\rho(x)} - \frac{e^{\mp izx}}{\rho(x)} \int_0^x e^{\pm i2zt} q(t)\, dt + \ell^d(n),$$
$$\int_0^x e^{\pm i2zt} q(t)\, dt = \ell^{d/2}(n), \tag{2.6}$$

$$f_\pm(z) = \frac{1}{\rho}\, (I - A_\mp(z) + A_\pm(z)\, A_\mp(z)$$
$$- A_\mp(z)\, A_\pm(z)\, A_\mp(z))\, e_\pm(z) + [\ell^d(n)]^2, \tag{2.7}$$

as $n \to \infty$, uniformly on bounded subsets of $[0, 1] \times \{|z - \pi n| \leqslant \pi\} \times H_C$.

Remark that $[\ell^d(n)]^2$ in (2.7) means that there exist two sequences $\beta = \{\beta_n(q)\}$, $\alpha = \{\alpha_n(q)\} \in \ell^2$ such that $\|\beta\|_d \leqslant 1$, $\|\alpha\|_d \leqslant 1$, and

$$f_\pm(z) = \frac{1}{\rho}\, (I - A_\mp(z) + A_\pm(z)\, A_\mp(z) - A_\mp(z)\, A_\pm(z)\, A_\mp(z))\, e_\pm(z)$$
$$+ \beta_n(q)\, \alpha_n(q)\, O(1),$$

as $n \to \infty$, uniformly on bounded subsets of $[0, 1] \times \{|z - \pi n| \leqslant \pi\} \times H_C$.

Below we need the following identities from [CM]

$$\varphi'(x, z, -q) = \rho(x)^2\, \vartheta(x, z, q),$$
$$\varphi(x, z, -q) = -\frac{\rho(x)^2}{z^2}\, \vartheta'(x, z, q), \qquad x \in [0, 1], \tag{2.8}$$

where $z \in \mathbb{C}$, $q \in H_C$, and now we prove some properties of the fundamental solutions.

LEMMA 2.2. *For each $x \in [0, 1]$ the functions $\varphi(x, z, q)$, $\vartheta(x, z, q)$ are entire on $\mathbb{C} \times H_C$. Moreover, φ and ϑ are analytic as a map from $\mathbb{C} \times H_C$ into $W^2_{2C}(0, 1)$ and the following estimates are fulfilled*

$$\max\{|z\varphi(x, z, q)|, |\varphi'(x, z, q)|, |\vartheta(x, z, q)|, |z^{-1}\vartheta'(x, z, q)|\} \leqslant e^{|\mathrm{Im}\, z|\, x + 2\, \|q\|_C}, \tag{2.9}$$

and their derivatives with respect to q have the forms

$$\partial\varphi(1, z, q) = 2\varphi'(t, z, q)\, G(1, t, z, q), \tag{2.10}$$

$$\partial\varphi'(1, z, q) = 2\varphi'(t, z, q)\, G_x(1, t, z, q), \tag{2.11}$$

$$\partial\vartheta(1, z, q) = 2\vartheta'(t, z, q)\, G(1, t, z, q), \tag{2.12}$$

$$\partial\vartheta'(1, z, q) = 2\vartheta'(t, z, q)\, G_x(1, t, z, q). \tag{2.13}$$

If the sequence q^ν converges weakly to q in H_C, as $\nu \to \infty$, then

$$\varphi(x, z, q^\nu) \to \varphi(x, z, q) \qquad \vartheta(x, z, q^\nu) \to \vartheta(x, z, q),$$
$$\varphi'(x, z, q^\nu) \to \varphi'(x, z, q), \qquad \vartheta'(x, z, q^\nu) \to \vartheta'(x, z, q),$$

uniformly on bounded subsets of $[0, 1] \times \mathbb{C}$. Moreover, for any fixed $d > 1$

$$\varphi(x, z, q) = \frac{1}{z\rho(x)}\left(\sin zx + \int_0^x \sin z(2t - x)\, q(t)\, dt + \ell^d(n)\right), \tag{2.14}$$

$$\varphi'(x, z, q) = \frac{1}{\rho(x)}\left(\cos zx - \int_0^x \cos z(2t - x)\, q(t)\, dt + \ell^d(n)\right), \tag{2.15}$$

as $n \to \infty$, uniformly on bounded subsets of $[0, 1] \times \{|z - \pi n| \leqslant \pi\} \times H_C$.

Proof. The proof repeats the corresponding arguments for the functions $f_\pm(x, z, q)$.

We now show (2.10)–(2.13). Let $q = q_0 + v$, where $q_0, v \in H_C$. Then using (2.4) we obtain

$$\varphi(1, z, q) = \varphi(1, z, q_0) - 2\int_0^1 G(1, t, zq_0), v(t)\, \varphi'(t, z, q)\, dt,$$

which yields (2.10), since the functions φ, ϑ are entire. The proof of (2.11)–(2.13) is similar.

Let the sequence q^ν converge weakly to q in H_C, as $\nu \to \infty$. Then using (2.1) and Lemma 2.1 we deduce that $\varphi'(x, z, q^\nu) \to \varphi'(x, z, q)$, $\vartheta'(x, z, q^\nu) \to \vartheta'(x, z, q)$, uniformly on bounded subsets of $[0, 1] \times \mathbb{C}$. Adding (2.8) we obtain the convergence $\varphi(x, z, q^\nu) \to \varphi(x, z, q)$, $\vartheta(x, z, q^\nu) \to \vartheta(x, z, q)$.

We prove (2.14)–(2.15). The relations (2.6) and (2.1) yield

$$\varphi'(x, z, q) = \frac{1}{2}(f_+(x, z, q) + f_-(x, z, q))$$

$$= \frac{1}{\rho(x)}\left(\cos zx - \int_0^x \cos z(2t - x)\, q(t)\, dt + \ell^d(n)\right).$$

Using (2.6), (2.1), and (2.8) we deduce that

$$\varphi(x, z, -q) = -\frac{\rho^2(x)}{z^2}\vartheta'(x, z, q) = \frac{1}{2iz}(f_+(x, z, q) - f_-(x, z, q))$$

$$= \frac{\rho(x)}{z}\left(\sin zx - \int_0^x \sin z(2t - x)\, q(t)\, dt + \ell^d(n)\right),$$

which gives (2.14). ∎

We need some results on Eq. (1.3) with potential shifted by parameter t. Let $\varphi(x, z, q, t)$, $\vartheta(x, z, q, t)$ be the solutions of the equation

$$-y'' - 2q(x + t)\, y' = z^2 y, \qquad z \in \mathbb{C}, \quad t \in \mathbb{R}, \tag{2.16}$$

satisfying $\varphi(0, z, q, t) = \vartheta(0, z, q, t) = 1$, and $\varphi(0, z, q, t) = \vartheta(0, z, q, t) = 0$. Remark that the Lyapunov function $\Delta(z, q, t)$ for (2.16) coincides with the Lyapunov function $\Delta(z, q)$ for $t = 0$. Any solution of the inhomogeneous equation

$$-y'' - 2q(x + t)\, y' - z^2 y = f, \tag{2.17}$$

has the form

$$y(x, t) = C_1 \varphi(x, z, q, t) + C_2 \vartheta(x, z, q, t)$$

$$-\int_0^x \varphi(x - s, z, q, s + t)\, f(s)\, ds. \tag{2.18}$$

Let $\chi_A(x)$ be the indicator function of the set $A \subset \mathbb{R}$.

Lemma 2.3. *For each $t \in \mathbb{R}$ the functions $\varphi(1, z, q, t)$, $\varphi'(1, z, q, t)$ are entire on $\mathbb{C} \times H_C$. Their gradients are given by the formulas*

$$
\begin{aligned}
&(\partial\varphi(1, z, q, t))(x) \\
&\quad = -2\chi_{[0,t]}(x)\,\varphi(t-x, z, q, 1+x)\,\varphi'(1+x-t, z, q, t) \\
&\qquad - 2\chi_{[t,1]}(x)\,\varphi(1+t-x, z, q, 1+x)\,\varphi'(x-t, z, q, t),
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
&(\partial\varphi'(1, z, q, t))(x) \\
&\quad = -2\chi_{[0,t]}(x)\,\varphi'(t-x, z, q, 1+x)\,\varphi'(1+x-t, z, q, t) \\
&\qquad - 2\chi_{[t,1]}(x)\,\varphi'(1+t-x, z, q, 1+x)\,\varphi'(x-t, z, q, t).
\end{aligned}
\tag{2.20}
$$

Proof. Let $q = q_0 + v$, $\varphi(s, t) = \varphi(s, z, q, t)$, $\varphi_0(s, t) = \varphi(s, z, q_0, t)$. Then using (2.18) we obtain

$$\varphi(1, t) = \varphi_0(1, t) - \int_0^1 \varphi_0(1-s, s+t)\, 2v(s+t)\, \varphi'(s, t)\, ds,$$

and hence

$$\varphi(1, t) = \varphi_0(1, t) - \int_t^{1+t} \varphi_0(1+t-x, x)\, 2v(x)\, \varphi'(x-t, t)\, ds,$$

which yields (2.19). The proof for $\varphi'(1, z, q, t)$ is similar. ∎

Let $q_n^2 \equiv q_{cn}^2 + q_{sn}^2$. Define the function $\Delta_2(z, q) = \Delta(z, q) - \cos z - \Delta_1(z, q)$ where

$$\Delta_1(z, q) = \int_0^1 q(t)\, dt \int_0^t q(s) \cos z(1-2t+2s)\, ds.$$

We need the properties of the function Δ.

LEMMA 2.4. (i) *If the sequence q^ν, $\nu \geqslant 1$, converges weakly to q in H, then $\Delta(z, q^\nu) \to \Delta(z, q)$ uniformly on bounded subsets of $\mathbb{C}$. The function $\Delta(\cdot, \cdot)$ is entire on $\mathbb{C} \times H_C$.*

(ii) *Its gradient is given by the formula*

$$(\partial\Delta(z, q))(x) = -\varphi'(1, z, q, x). \tag{2.21}$$

In particular, if $|\gamma_n| = 0$, then $\partial\Delta(z, q) = 0$ at $z = m_n(q)$.

(iii) *Moreover, for any fixed* $d > 1$, *the following asymptotic estimates are fulfilled,*

$$\Delta_1(z, q) = \ell^d(n),$$

$$\Delta_2(z, q) = (\ell^d(n))^2, \qquad \textit{as} \quad |z| \to \infty, \quad |z - \pi n| \leqslant \pi, \tag{2.22}$$

$$\Delta_1(\pi n, q) = \frac{(-1)^n}{2} q_n^2, \tag{2.23}$$

and

$$(\partial \Delta(z, q))(t) = \int_0^1 q(x+t) \cos z(2x-1)\, dx + \ell^1(n),$$

$$|z - \pi n| \leqslant \pi, \quad n \in \mathbb{Z}, \tag{2.24}$$

$$(\partial \Delta(z, q))(t) = (-1)^n\, d_q q_n^2 + \ell^1(n), \qquad \textit{if} \quad z - \pi n = \ell^2(n), \tag{2.25}$$

as $|z| \to \infty$. *These estimates are infinitely differentiable with respect to* z *and are satisfied uniformly on bounded subsets of* $[0, 1] \times H_C$.

(iv) *For fixed* $z \in \mathbb{C}$ *the function* $\Delta(z, q)$ *is even with respect to* $q \in H_C$ *and*

$$\Delta(z, -q) = \Delta(z, q) = \tfrac{1}{2}[\varphi'(1, z, q) + \varphi'(1, z, -q)], \qquad q \in H_C, \tag{2.26}$$

$$\dot{\Delta}(z, q) = -z \int_0^1 \varphi(1, z, q, t)\, dt. \tag{2.27}$$

Proof. (i) Using the properties of the fundamental solutions (Lemma 2.2) we obtain the results for the Lyapunov function Δ stated in (i).

(ii) Let $\varphi(x, t) = \varphi(x, z, q, t)$, $\varphi(x) = \varphi(x, 0)$ and $\vartheta(x, t) = \varphi(x, z, q, t)$, $\vartheta(x) = \varphi(x, 0)$. We have the identities

$$\varphi(1+t) = \varphi(1)\, \vartheta(t) + \varphi'(1)\, \varphi(t), \qquad \vartheta(1+t) = \vartheta(1)\, \vartheta(t) + \vartheta'(1)\, \varphi(t). \tag{2.28}$$

It is easy to find the equality

$$\varphi(x, t) = \rho(t)^2\, (\varphi(x+t)\, \vartheta(t) - \vartheta(x+t)\, \varphi(t)), \tag{2.29}$$

since $\varphi(0, t) = 0$, $\varphi'(0, t) = 1$. Substituting (2.11)–(2.12) into (1.4) we have

$$\begin{aligned}(\partial \Delta(z, q))(t) &= \rho^2(t)[\varphi'(t)(\vartheta'(1)\,\varphi(t) - \varphi'(1)\,\vartheta(t)) \\ &\quad + \vartheta'(t)(\vartheta(1)\,\varphi(t) - \varphi(1)\,\vartheta(t))] \\ &= \rho^2(t)[\varphi(t)(\vartheta'(1)\,\varphi'(t) + \vartheta(1)\,\vartheta'(t)) \\ &\quad - \vartheta(t)(\varphi'(1)\,\varphi'(t) + \varphi(1)\,\vartheta'(t))]\end{aligned}$$

and (2.28)–(2.29) imply

$$(\partial \Delta(z, q))(t) = \rho^2(t)[\varphi(t)\,\vartheta'(1+t) - \vartheta(t)\,\varphi'(1+t)] = -\varphi'(1, t).$$

(iii) The asymptotic estimates (2.22) and identity (2.23) were proved in [K1]. Using (2.21) and (2.15) we get (2.24)–(2.25).

(iv) Using the identity (2.8) we obtain $\Delta(z, q) = \frac{1}{2}[\varphi'(1, z, q,) + \varphi'(1, z, -q)]$. The function Δ is even with respect to q, since the function $\varphi'(1, z, q)$ is real analytic with respect to q.

By (2.4), the solution of the equation

$$-\dot{\vartheta}'' - 2q\dot{\vartheta} - z^2\dot{\vartheta} = 2z\vartheta$$

has the form

$$\dot{\vartheta}(x) = 2z \int_0^x \rho(t)^2\, \vartheta(t)(\vartheta(x)\,\varphi(t) - \vartheta(t)\,\varphi(x))\, dt,$$

and then

$$\dot{\vartheta}(1) = 2z \int_0^1 \rho(t)^2\, \vartheta(t)(\vartheta(1)\,\varphi(t) - \vartheta(t)\,\varphi(1))\, dt. \tag{2.30}$$

The same consideration yields

$$\dot{\varphi}'(1) = 2z \int_0^1 \rho(t)^2\, \varphi(t)(\vartheta'(1)\,\varphi(t) - \vartheta(t)\,\varphi'(1))\, dt, \tag{2.31}$$

summing we get

$$\begin{aligned}\dot{\Delta} = z \int_0^1 \rho(t)^2\, &[\vartheta(t)(\vartheta(1)\,\varphi(t) - \vartheta(t)\,\varphi(1)) \\ &+ \varphi(t)(\vartheta'(1)\,\varphi(t) - \vartheta(t)\,\varphi'(1))]\, dt,\end{aligned}$$

and using (2.29) we obtain

$$\dot{\Delta} = z \int_0^1 \rho(t)^2 \left[\varphi(t)\, \vartheta(1+t) - \vartheta(t)\, \varphi(1+t)\right] dt = -z \int_0^1 \varphi(1, t)\, dt. \quad \blacksquare$$

Introduce the contour $C_n(r) = \{z\colon |z - \pi n| = r\}$, $n \geqslant 0$, $r > 0$. Below we shall use the following results on the Dirichlet problem for Eq. (1.3) on the interval $[0, 1]$.

LEMMA 2.5. (i) *Let $q \in H_{\mathbb{C}}$ and $\varepsilon_q = 8^{-2} \exp(-4\, \|q\|)$. Then for each integer $N > N_q$ and any $p \in B_{\mathbb{C}}(q, \varepsilon_q)$ the function $\varphi(1, z, p)$ has exactly $2N$ roots, counted with multiplicities, in the disc $\{z\colon |z| < \pi(N + (1/2))\}$ and for each $|n| > N$, exactly one simple root in the domain $\{z\colon |z - \pi n| < 1\}$. There are no other roots.*

(ii) *Each function $m_n(\cdot)$, $n \geqslant 1$, is compact and real analytic on H. Its gradient is given by the formula*

$$(d_q m_n(q))(x) = \frac{\rho(x)^2}{2m_n(q)\, \|\varphi(\cdot, m_n(q), q)\|_{\rho^2}^2} (\varphi(x, m_n(q), q)^2)', \qquad (2.32)$$

where

$$\|\varphi(\cdot, m_n(q), q)\|_{\rho^2}^2 = \frac{\varphi'(1, m_n(q), q)\, \dot{\varphi}(1, m_n(q), q)}{2m_n(q)} > 0. \qquad (2.33)$$

Moreover, for any fixed $d > 1$ the following asymptotic estimates are fulfilled,

$$m_n(q) = \pi n - q_{sn} + \ell^d(n), \qquad (2.34)$$

$$(d_q m_n(q))(x) = -\sin 2\pi n x + \ell^2(n), \qquad (2.35)$$

as $n \to \infty$, uniformly on $[0, 1] \times B_{\mathbb{C}}(p, \varepsilon_p)$ for each fixed $p \in H$.

Proof. (i) Let $M > N$ be another integer. Consider the contours $C_0(\pi(N + (1/2)))$, $C_0(\pi(M + (1/2)))$, $C_n(1)$, $|n| > N$. Recall that for q we defined the functions $\tilde{q}$, $\tilde{V}$ and the integer $m \geqslant 1$ (see (1.18)–(1.19)) such that $\|q - \tilde{q}\| \leqslant \varepsilon$. We have the identity

$$\varphi(1, z, \tilde{q}) = y_2(1, z, \tilde{V}),$$

where $y_2(x, z, \tilde{V})$ is the solution of the equation

$$-y_2'' + \tilde{V} y_2 = z^2 y_2, \qquad y_2(0, z, \tilde{V}) = 0, \qquad y_2'(0, z, \tilde{V}) = 1,$$

where $\tilde{V}$ is defined by (1.19). Using the estimates in [PT] on y_2, it is easy to get

$$\left| y_2(1, z, \tilde{V}) - \frac{\sin z}{z} \right| \leqslant \frac{A}{|z|} \exp\{A + |\operatorname{Im} z|\},\qquad A = \frac{1}{|z|} \int_0^1 |\tilde{V}(x)|\, dx. \tag{2.36}$$

Then (1.20) implies

$$A \leqslant \frac{\|q\|}{|z|} (\|q\| + 2\pi m) < 4\varepsilon, \qquad \varepsilon = \varepsilon_q,$$

where $|z| \geqslant \pi(N + (1/2))$. We have the equality

$$\varphi(1, z, p) - \varphi(1, z, \tilde{q}) = \int_0^1 (\partial\varphi(1, z, \tilde{q} + tv), v)\, dt, \qquad v = p - \tilde{q}, \tag{2.37}$$

where $(\cdot, \cdot)$ is the scalar product in H_C. Combining the identities (2.19) with the estimates (2.9) yields

$$\begin{aligned} |\partial\varphi(1, z, \tilde{q} + tv)| &\leqslant \frac{2}{|z|} e^{|\operatorname{Im} z| + 4\|\tilde{q} + tv\|} \\ &\leqslant \frac{2}{|z|} e^{|\operatorname{Im} z| + 4\|q\| + 8\varepsilon}, \qquad t \in (0, 1), \end{aligned} \tag{2.38}$$

since $\|v\| \leqslant \|p - q\| + \|q - \tilde{q}\| \leqslant 2\varepsilon$. The substitution of this estimate into (2.37) implies

$$|\varphi(1, z, p) - \varphi(1, z, \tilde{q})| \leqslant \frac{4\varepsilon}{|z|} e^{|\operatorname{Im} z| + 4\|q\| + 8\varepsilon}. \tag{2.39}$$

Using (2.36) and (2.39) we obtain

$$\begin{aligned} \left| \varphi(1, z, p) - \frac{\sin z}{z} \right| &\leqslant |\varphi(1, z, p) - \varphi(1, z, \tilde{q})| + \left| \varphi(1, z, \tilde{q}) - \frac{\sin z}{z} \right| \\ &\leqslant \frac{4\varepsilon}{|z|} e^{|\operatorname{Im} z| + 8\varepsilon}(1 + e^{4\|q\|}). \end{aligned}$$

Therefore, the simple estimate $\exp|\operatorname{Im} z| < 4\,|\sin z|$ on all contours (see [PT] yields

$$\left|\varphi(1, z, p) - \frac{\sin z}{z}\right| \leqslant C\,\frac{|\sin z|}{|z|}, \qquad C = 16\varepsilon e^{8\varepsilon}(1 + e^{4\,\|q\|}),$$

and $C<1$ on all contours. Hence, by Rouchés theorem, $\varphi(1, z, r)$ has as many roots, counted with multiplicities, as $\sin z/z$ in each of the bounded domains and the remaining unbounded region. Since $\sin z/z$ has only simple roots at πn, $n \neq 0$, and since $M > N$ can be chosen arbitrarily large, the point (i) of Lemma 2.5 follows.

(ii) First we prove (2.33) which shows that the zero $m_n(q)$ is simple for $q \in H$. In fact, we repeat the case of the potential from [PT]. Differential of Eq. (1.3) with respect to z implies

$$-\dot{\varphi}'' - 2q\dot{\varphi}' = 2z\varphi + z^2\dot{\varphi}.$$

Multiplying this equation by φ, the original equation by $\dot{\varphi}$ and taking the difference we deduce that

$$2z\varphi^2 = (\dot{\varphi}\varphi'' - \dot{\varphi}''\varphi) + 2q(\dot{\varphi}\varphi' - \dot{\varphi}'\varphi)$$

and

$$2z\rho^2\varphi^2 = \rho^2[\dot{\varphi}, \varphi]' + 2q\rho^2[\dot{\varphi}, \varphi] = (\rho^2[\dot{\varphi}, \varphi])'.$$

Let $m_n = m_n(q)$. Then

$$\begin{aligned} 2m_n\,\|\varphi(\cdot, m_n)\|^2_{\rho^2} &= \int_0^1 \rho(x)^2\,\varphi(x, m_n)^2\,dx \\ &= \int_0^1 (\rho(x)^2\,[\dot{\varphi}(x, m_n), \varphi(x, m_n)])'\,dx \\ &= \dot{\varphi}(1, m_n)\,\varphi'(1, m_n), \end{aligned}$$

since $\varphi(0, z)$ and $\dot{\varphi}(0, z)$ vanish for all z and $\varphi(1, z)$ vanishes for a Dirichlet eigenvalue $z = m_n$. The integral is equal to $\|\varphi(\cdot, z)\|_{\rho^2}$ since φ is real for real z.

In order to prove compactness of $m_n(q)$ suppose that the sequence q^ν, $\nu \geqslant 1$, converges weakly to q in H. For small $\delta > 0$ we introduce the intervals

$$I_n = [m_n(q) - \delta, m_n(q) + \delta] \subset (0, \infty), \qquad 1 \leqslant n \leqslant N.$$

If δ is sufficiently small, then these intervals are all disjoint. The function $\varphi(1, z, q)$ changes sign on each of them and $|\dot{\varphi}(1, z, q)| > 2B$, $z \in \bigcup I_n$, for some $B > 0$ since $m_n(q)$ is a simple root.

As $\nu \to \infty$, the functions $\varphi(1, z, q^\nu) \to \varphi(1, z, q)$ and $\dot{\varphi}(1, z, q^\nu) \to \dot{\varphi}(1, z, q)$ converge uniformly on $\bigcup I_n$ by Lemma 2.2. Hence, for sufficiently large ν, the functions $\varphi(1, z, q^\nu)$ also change sign and $|\dot{\varphi}(1, z, q^\nu)| > B$ on $\bigcup I_n$, so they must all have one root in each of these intervals, for ν sufficiently large. Hence $\varphi(1, z, q^\nu)$ has exactly one root $m_n(q^\nu)$ in each segment I_n, $1 \leqslant n \leqslant N$. This yields $|m_n(q^\nu) - m_n(q)| \leqslant \delta$, $1 \leqslant n \leqslant N$, for all sufficiently large ν. It follows that $m_n(q^\nu) \to m_n(q)$, as $\nu \to \infty$, since N and $\delta > 0$ were arbitrary. Thus, $m_n(q)$ is a compact functions of q.

To prove real analyticity, we fix $r \in H$. Then $\dot{\varphi}(1, z_n(r), r) \neq 0$. Now, the implicit function theorem guarantees the existence of a unique continuous function $\tilde{m}_n$ defined on some small neighborhood $W \subset H$ of r and such that

$$\varphi(1, \tilde{m}_n(q), q) = 0, \qquad \tilde{m}_n(r) = m_n(r)$$

on W. Furthermore, $\tilde{m}_n(q)$ is real analytic. On the other hand, $m_n(q)$ is also a continuous function on W satisfying $\varphi(1, m_n(q), q) = 0$. Therefore, by uniqueness, $\tilde{m}_n(q) = m_n(q)$ on W, and so $m_n(q)$ is real analytic.

To calculate the gradient, we observe that $\varphi(1, m_n(q), q) = 0$. Hence

$$\begin{aligned} 0 &= d_q\{\varphi(1, m_n(q), q)\} \\ &= \dot{\varphi}(1, m_n(q), q)\, d_q m_n + \partial\varphi(1, z, q), \qquad z = m_n(q). \end{aligned}$$

By (2.10), the second term has the form

$$(\partial\varphi(1, z, q))(x) = 2\rho(x)^2\, \vartheta(1, z, q)(\varphi^2(x, z, q))'.$$

The Wronskian identity $\vartheta(1, z, q)\, \varphi'(1, z, q) = 1$ yields

$$\begin{aligned} (d_q m_n)(x) &= -\frac{2\rho(x)^2\, \vartheta(1, z, q)(\varphi^2(x, z, q))'}{\dot{\varphi}(1, z, q)} \\ &= -\frac{2\rho(x)^2\, (\varphi^2(x, z, q))'}{\dot{\varphi}(1, z, q)\, \varphi'(1, z, q)} \end{aligned}$$

which together with (2.33) implies (2.32).

Relation (2.14) yields

$$z\varphi(1, z, -q) = \sin z - \int_0^1 q(x) \sin z(2x - 1)\, dx + \ell^d(n),$$

$$|z - \pi n| \leqslant 1, \quad n \to \infty.$$

Hence

$$0 = m_n\varphi(1, m_n, q) = \sin m_n + \int_0^1 q(x) \sin 2m_n x \, dx + \ell^d(n),$$

which implies (2.34).

The asymptotic estimate (2.14) gives

$$\|\varphi(\cdot, m_n, q)\|_{\rho^2}^2 = \frac{1}{m_n^2} \int_0^1 (\sin^2 xm_n + \ell^2(n)) \, dx, \qquad n \to \infty,$$

and then

$$\|\varphi(\cdot, m_n, q)\|_{\rho^2}^2 = \frac{1}{m_n^2} \left[\frac{1}{2} - \frac{\sin 2m_n}{4m_n} + \ell^2(n) \right] = \frac{1}{m_n^2} (1 + \ell^2(n)). \tag{2.40}$$

Using (2.14)–(2.15) and (2.40) we obtain

$$\begin{aligned} d_q m_n &= \rho^2(x) \frac{2\varphi'(x, m_n, q) \, \varphi(x, m_n, q)}{2m_n \, \|\varphi(\cdot, m_n, q)\|_{\rho^2}^2} \\ &= -2 \frac{[\cos xm_n + \ell^2(n)][\sin xm_n + \ell^2(n)] \, m_n^2}{\|\varphi(\cdot, m_n, q)\|_{\rho^2}^2} \\ &= -\sin 2xm_n + \ell^2(n), \end{aligned}$$

which implies (2.35). ∎

3. MAPPINGS

In order to study other mappings, we must consider the function $z_n(q)$. Recall that N_q is defined by (1.18).

LEMMA 3.1. (i) *Let $q \in H_C$ and $\varepsilon_q = 8^{-2}\exp(-4\,\|q\|)$. Then for any integer $N \geqslant N_q$ and for each $p \in B_C(q, \varepsilon_q)$ the function $\dot{\Delta}(z, p)$ has exactly $2N+1$ roots, counted with multiplicities, in the disc $\{z\colon |z| < \pi(N + (1/2))\}$ and for each $|n| > N$, it has exactly one simple root in the domain $\{z\colon |z - \pi n| < 1\}$. There are no other roots.*

(ii) *Each function $z_n(\cdot)$, $n \geqslant 1$, is compact and real analytic on H. Its gradient is given by the formula*

$$(d_q z_n(q))(x) = -\frac{(\partial \dot{\Delta})(x, z_n(q), q)}{\ddot{\Delta}(z_n(q), q)}. \tag{3.1}$$

Moreover, for any fixed $d > 1$ *the following asymptotic estimates are fulfilled,*

$$z_n(q) = \pi n + \ell^d(n), \qquad n \to \infty, \tag{3.2}$$

$$(d_q z_n(q))(x) = \ell^2(n), \qquad n \to \infty, \tag{3.3}$$

uniformly on $[0, 1] \times B_C(p, \varepsilon_p)$ *for each fixed* $p \in H$.

Proof. (i) Fix N and let $M > N$ be another integer. Consider the contours $C_0(\pi(N + (1/2)))$, $C_0(\pi(M + (1/2)))$, $C_n(1)$, $|n| > N$. Recall that for q we defined the functions $\tilde{q}$, $\tilde{V}$ (see (1.18)–(1.19)) such that $\|q - \tilde{q}\| \leqslant \varepsilon$ and

$$\Delta(z, \tilde{q}) = D(z, \tilde{V}),$$

where $D(z, \tilde{V})$ is the Lyapunov function for the Hill operator. Using the estimates in [PT] it is easy to get

$$|\dot{D}(z, \tilde{V}) - \sin z| \leqslant A \exp\{A + |\operatorname{Im} z|\}, \qquad A = \frac{1}{|z|} \int_0^1 |\tilde{V}(x)|\, dx. \tag{3.4}$$

If we take z such that $|z| \geqslant \pi(N + (1/2))$, then (1.20) implies

$$A \leqslant \frac{\|q\|}{|z|} (\|q\| + 2\pi m) < 4\varepsilon.$$

Hence, using (2.27) we obtain

$$|\dot{\Delta}(z, p) - \dot{\Delta}(z, \tilde{q})| = \left| z \int_0^1 (\varphi(1, z, q, t) - \varphi(1, z, \tilde{q}, t))\, dt \right|, \tag{3.5}$$

and

$$\varphi(1, z, p, t) - \varphi(1, z, \tilde{q}, t)$$
$$= \int_0^1 (\partial \varphi(1, z, \tilde{q} + yv, t), v)\, dy, \qquad v = p - \tilde{q}, \tag{3.6}$$

where $(\cdot, \cdot)$ is the scalar product in H_C. The identities (2.19) and the estimate (2.9) yield

$$|\partial \varphi(1, z, \tilde{q} + yv, t)| \leqslant \frac{2}{|z|} e^{|\operatorname{Im} z| + 4\|\tilde{q} + yv\|} \leqslant \frac{2}{|z|} e^{|\operatorname{Im} z| + 4\|q\| + 8\varepsilon}, \tag{3.7}$$

since $\|v\| \leqslant \|p-q\| + \|q-\tilde{q}\| \leqslant 2\varepsilon$. Substituting this estimate into (3.5) we have

$$|\dot{\Delta}(z, p) - \dot{\Delta}(z, \tilde{q})| \leqslant 4\varepsilon e^{|\mathrm{Im}\, z| + 4\, \|q\| + 8\varepsilon}. \tag{3.8}$$

Using (3.4) and (3.8) we deduce that

$$\begin{aligned} |\dot{\Delta}(z, p) + \sin z| &\leqslant |\dot{\Delta}(z, p) - \dot{\Delta}(z, \tilde{q})| + |\dot{\Delta}(z, \tilde{q}) + \sin z| \\ &\leqslant 4\varepsilon e^{|\mathrm{Im}\, z| + 8\varepsilon}(1 + e^{4\, \|q\|}). \end{aligned}$$

Therefore, the simple estimate $\exp |\mathrm{Im}\, z| < 4\, |\sin z|$ on all contours (see [PT]) yields

$$|\dot{\Delta}(z, p) + \sin z| \leqslant C\, |\sin z|, \qquad C = 16\varepsilon e^{8\varepsilon}(1 + e^{4\, \|q\|}).$$

By assumption, $C < 1$ on all contours. Hence, by Rouché's theorem, $\dot{\Delta}(\cdot, p)$ has as many roots, counted with multiplicities, as $\sin z$ in each of the bounded domains and the remaining unbounded region. Since $\sin z$ has only simple roots πn, $n \in \mathbb{Z}$, and since $M > N$ can be chosen arbitrarily large, the point (i) of Lemma 3.1 follows.

(ii) In order to prove compactness of $z_n(q)$, suppose the sequence q^ν, $\nu \geqslant 1$, converges weakly to q in H. Let $z_n^0(q)$, $n \geqslant 1$, be the zeros of the function $\Delta(z, q)$, $z > 0$. Define the segments $I_n^0 = [z_n^0(q), z_{n+1}^0(q)]$, $n \geqslant 1$. In each I_n^0, $n \geqslant 1$, there exists exactly one zero $z_n(q)$ of the function $\dot{\Delta}(z, q)$, and

$$z_n^0(q) < z_n(q) < z_{n+1}^0(q)], \qquad n \geqslant 1, \tag{3.9}$$

since the function $\Delta(\cdot, q)$ is entire with real zeroes (see [K1]). Using Theorem 2.1 we obtain $\Delta(z, q^\nu) \to \Delta(z, q)$ and $\dot{\Delta}(z, q^\nu) \to \dot{\Delta}(z, q)$ as $\nu \to \infty$, uniformly on bounded subset of $\mathbb{C}$. Then by (3.9), in each I_n^0, $1 \leqslant n \leqslant N$, for any $N \geqslant 2$, there exists exactly one zero $z_n(q^\nu)$ of the function $\dot{\Delta}(z, q^\nu)$ for large ν. For small $\delta > 0$ we introduce the intervals

$$I_n = [z_n(q) - \delta, z_n(q) + \delta] \subset I_n^0, \qquad 1 \leqslant n \leqslant N.$$

If δ is sufficiently small, then these intervals are all disjoint and $I_n \subset I_n^0$, $1 \leqslant n \leqslant N$. The function $\dot{\Delta}(z, q)$ changes sign on each of them, since $z_n(q)$ is a simple root.

As $\nu \to \infty$, the functions $\dot{\Delta}(z, q^\nu)$ converge to $\dot{\Delta}(z, q)$ uniformly on $\bigcup I_n$ by Lemma 2.4. Hence, for sufficiently large ν, they also change sign on $\bigcup I_n$, so they must all have at least one root in each of these intervals. The functions $|\Delta(z, q)| \geqslant 1/2$ and $|\Delta(z, q^\nu)| \geqslant 1/4$ for any $z \in \bigcup I_n$ and for sufficiently large ν. Hence by (3.9), $\dot{\Delta}(z, q^\nu)$ has exactly one root $z_n(q^\nu)$ in each segment I_n, $1 \leqslant n \leqslant N$. This yields $|z_n(q^\nu) - z_n(q)| < \delta$, $1 \leqslant n \leqslant N$, for all sufficiently large v. It follows that $z_n(q^v) \to z_n(q)$, as $v \to \infty$, since N and $\delta > 0$ were arbitrary. Thus, $z_n(q)$ is a compact functions of q.

To prove real analyticity, we fix $r \in H$. Then $\dot{\Delta}(z_n(r), r) = 0$ and $\ddot{\Delta}(z_n, (r), r) \neq 0$. Now, the implicit function theorem guarantees the existence of a function $\tilde{z}_n$ defined on some small neighborhood $W \subset H$ of r and such that

$$\dot{\Delta}(\tilde{z}_n(q), q) = 0, \qquad \tilde{z}_n(r) = z_n(r)$$

on W. Furthermore, $\tilde{z}_n(q)$ is real analytic. On the other hand, $z_n(q)$ is also a continuous function on W satisfying $\dot{\Delta}(z_n(q), q) = 0$. Therefore, by uniqueness, $\tilde{z}_n(q) = z_n(q)$ on W, and so $z_n(q)$ is real analytic.

To calculate the gradient, we observe that $\dot{\Delta}(z_n(q), q) = 0$. Hence

$$0 = d_q\{\dot{\Delta}(z_n(q), q)\} = \ddot{\Delta}(z_n(q), q)\, d_q z_n + \partial \dot{\Delta}(z, q),$$

which implies (3.1).

Using (i), we obtain $|z_n(q) - \pi n| \leqslant 1$. We can improve (i). The asymptotic estimates (2.22) yield $\dot{\Delta}(z_n(q), q) = -\sin z_n(q) + \ell^d(n)$. Then $\sin(z_n(q) - \pi n) = \ell^d(n)$ which implies (3.2). Hence from (2.22), (2.25) we conclude

$$d_q z_n(q) = -\frac{\partial \dot{\Delta}(z_n(q), q)}{\ddot{\Delta}(z_n(q), q)} = \frac{\ell^2(n)}{\cos z_n(q) + \ell^2(n)} = \ell^2(n). \quad \blacksquare$$

We need the following results concerning the function $r_n = h_n^2$.

Lemma 3.2. *Each function $r_n(\cdot)$, $n \geqslant 1$, is compact and real analytic on H. Its gradient is given by the formula*

$$(d_q r_n(q))(x) = (-1)^n \frac{(\partial \Delta(z_n(q), q))(x)}{(d \cosh \sqrt{r_n}/dr_n)}, \qquad n \geqslant 1. \tag{3.10}$$

Moreover, for any fixed $d > 1$ the following estimates are fulfilled,

$$r_n(q) = q_n^2 + (\ell^d(n))^3, \qquad n \to \infty, \tag{3.11}$$

$$(d_q r_n(q))(w) = d_q q_n^2 + \ell^d, \qquad n \to \infty, \tag{3.12}$$

uniformly on $[0, 1] \times B_C(p, \varepsilon_p)$ for each fixed $p \in H$.

Proof. Since $\Delta(z_n(q), q)$ is compact, so is $r_n(q)$. In order to prove real analyticity, we fix $v \in H$. Then $(-1)^n\, \Delta(z_n(v), v) = \cosh \sqrt{r_n}$ and $(\cosh \sqrt{r_n})'_{r_n} \neq 0$. Now, the implicit function theorem guarantees the existence of a unique continuous function $\tilde{r}_n$ defined on some small neighborhood $W \subset H$ of v and such that

$$(-1)^n \Delta(z_n(q), q) = \cosh \sqrt{\tilde{r}_n(q)}, \qquad \tilde{r}_n(v) = r_n(v),$$

on W. Furthermore, $\tilde{r}_n(q)$ is real analytic. On the other hand, $r_n(q)$ is also a continuous function on W satisfying $(-1)^n \Delta(z_n(q), q) = \cosh \sqrt{r_n(q)}$. Therefore, by uniqueness, $\tilde{r}_n(q) = r_n(q)$ on W, and so $r_n(q)$ is real analytic.

To calculate the gradient, we observe that $(-1)^n \Delta(z_n(q), q) = \cosh \sqrt{r_n(q)}$. Hence the implicit function theorem yields

$$(-1)^n (d \cosh \sqrt{r_n}/dr_n)\, d_q r_n = \partial \Delta(z_n(q), q) + \dot{\Delta}(z_n(q), q)\, d_q z_n(q),$$

and the identity $\dot{\Delta}(z_n(q), q) = 0$ implies (3.10).

We need a simple estimate concerning the analytic function $w = w(z) = \cosh \sqrt{z} - 1$, where $z \in \mathbb{C}$. There exists a domain $B \subset \mathbb{C}$, $0 \in B$, such that the new analytic function $w_1(z) \equiv w(z)$, $z \in B$, has an inverse function $g(w)$, analytic in $|w| < 2$, and the following estimates are fulfilled:

$$z = g(w) = 2w(1 - w/6 + \cdots), \qquad w \to 0. \tag{3.13}$$

Now we let $n \to \infty$. The function $t_n(q) \equiv (-1)^n \Delta(z_n(q), q) - 1$ is real analytic and (3.2), (2.22) imply the estimates

$$t_n = [(-1)^n \cos z_n + q_n^2/2 + (\ell^d(n))^2] - 1 = \frac{q_n^2}{2} + (\ell^d(n))^2 \tag{3.14}$$

uniformly on $B_C(p, \varepsilon_p)$. Hence the function $t_n(r_n) = \cosh \sqrt{r_n} - 1$ has an inverse $r_n = g(t_n)$ for $|t_n| < 2$, and (3.13)–(3.14) yield

$$r_n = 2t_n(1 - t_n/6 + \cdots) = q_n^2 + (\ell^d(n))^2.$$

Substituting the asymptotic estimates (3.2), (3.11), (2.25) into (3.10) we obtain

$$d_q r_n = (-1)^n \frac{\partial \Delta(z_n(q), q)}{(\sinh \sqrt{r_n}/2\sqrt{r_n})} = \frac{d_q q_n^2 + \ell^d(n)}{1 + O(r_n)} = d_q q_n^2 + \ell^d(n),$$

as $n \to \infty$, uniformly on $[0, 1] \times B_C(p, \varepsilon_p)$. ∎

4. ANALYTIC ISOMORPHISM

We need the following result concerning the mapping $h_{cn}(\cdot)$,

LEMMA 4.1. *Each function $h_{cn}(\cdot)$, $n \geqslant 1$, is compact and real analytic on H. Its gradient is given by the formulas*

$$d_q h_{cn} = -\frac{1}{\varphi'(1, m_n, q)} \{\dot{\varphi}'(1, m_n, q)\, d_q m_n(q) + \partial\varphi'(1, m_n, q)\}, \qquad n \geqslant 1, \tag{4.1}$$

$$d_q h_{cn} = \frac{(-1)^n}{\sinh h_{cn}} \{\dot{\Delta}(m_n(q), q)\, d_q m_n(q) + \partial\Delta(\mu_n(q), q)\}, \qquad h_{cn} \neq 0. \tag{4.2}$$

Moreover, for any fixed $d > 1$ the following estimates are fulfilled,

$$h_{cn}(q) = q_{cn} + \ell^d(n), \tag{4.3}$$

$$(d_q h_{cn}(q))(x) = \cos 2\pi n x + \ell^2(n), \tag{4.4}$$

as $n \to \infty$, uniformly on subsets of $[0, 1] \times B_C(p, \varepsilon_p)$ for each fixed $p \in H$.

Proof. $\varphi'(1, z, q)$ and $m_n(q)$ are compact, real analytic functions of (z, q) and q, respectively. Then $h_{cn}(q) = -\log(-1)^n \varphi'(1, m_n(q), q)$ is also compact and real analytic since $\varphi'(1, m_n(q), q)$ never vanishes. The differentiation rule yields,

$$d_q h_{cn}(q) = -\frac{1}{\varphi'(1, z, q)} [\dot{\varphi}'(1, z, q)\, d_q m_n(q) + \partial\varphi'(1, z, q)], \quad \text{at } z = m_n(q).$$

Using (1.7), Lemmas 2.4–2.5, and the implicit function theorem we deduce that

$$\begin{aligned}(\sinh h_{cn})\, d_q h_{cn} \\ &= (-1)^n [\partial\Delta(m_n(q), q) + \dot{\Delta}(m_n(q), q)\, d_q m_n(q)], \qquad h_{cn} \neq 0. \end{aligned} \tag{4.5}$$

Then Lemma 2.5 and (2.15) yield

$$\begin{aligned} e^{-h_{cn}} &= (-1)^n \varphi'(1, m_n(q), q) \\ &= (-1)^n \left(\cos m_n(q) - \int_0^1 q(x) \cos m_n(q)(2x-1)\, dx + \ell^d(n) \right) \\ &= \cos(m_n(q) - \pi n) - q_{cn} + \ell^d(n) = 1 - q_{cn} + \ell^d(n), \end{aligned}$$

which gives (4.3).

Using (2.20), (2.15), (2.34) we get

$$\begin{aligned}\partial\varphi'(1,z,q) &= -2\varphi'(1-x,z,q,x)\,\varphi'(x,z,q)\\ &= -2\cos z(1-x)\cos zx + \ell^2(n)\\ &= -(-1)^n\cos 2\pi nx + \ell^2(n), \qquad z = m_n(q),\end{aligned}$$

and

$$\dot{\varphi}'(1,m_n,q) = -\sin m_n + \ell^2(n) = \ell^2(n).$$

Substituting these asymptotics and (2.34)–(2.35) into (4.1) we obtain (4.4). ∎

Now we consider the function $h_{sn}(\cdot)$.

Lemma 4.2. *Each function $h_{sn}(\cdot)$, $n \geqslant 1$, is compact and real analytic on H. There exists a compact, real analytic and positive function $b_n(\cdot)$ on H such that*

$$h_{sn}(q) = (z_n(q) - m_n(q))\,b_n(q), \qquad q \in H, \tag{4.6}$$

and if $h_{sn}(v) = 0$ for some $v \in H$, then

$$d_q h_{sn}(v) = b_n(v)(d_q z_n(v) - d_q m_n(v)). \tag{4.7}$$

Moreover, for any fixed $d > 1$, the following estimate are fulfilled,

$$b_n(q) = 1 + \ell^2(n), \qquad d_q b_n(q) = \ell^2(n), \tag{4.8}$$

$$h_{sn}(q) = q_{sn} + \ell^d(n), \tag{4.9}$$

$$(d_q h_{sn}(q))(x) = \sin 2\pi nx + \ell^2(n), \tag{4.10}$$

as $n \to \infty$, uniformly on $[0,1] \times B_C(p, \varepsilon_p)$ for each fixed $p \in H$.

Proof. Introduce the value $r_{cn} = h_{cn}^2$ and the function $f_n(q) = f(r_n(q), r_{cn}(q))$, $q \in H$, where

$$f(x,y) = 2\left[\frac{1}{2} + \frac{1}{4!}(x+y) + \frac{1}{6!}(x^2 + xy + y^2) + \cdots\right].$$

f is an entire function of two parameters x, y and positive if $x \geqslant 0$, $y \geqslant 0$. Then $f_n(q)$ is compact, real analytic, and positive on H. Using (4.3)–(4.4) and (3.11)–(3.12) we obtain

$$f_n(q) = 1 + \ell^1(n), \qquad d_q f_n(q) = \ell^2(n), \tag{4.11}$$

as $n \to \infty$, uniformly on $[0,1] \times B_{\mathbb{C}}(p, \varepsilon_p)$ for each fixed $p \in H$. Fixing $q \in H$, we apply Taylor's formula for $\Delta(z) = \Delta(z, q)$, $m_n = m_n(q)$, $z_n = z_n(q)$, $\zeta_n = m_n - z_n$, with remainder in integral form, at $z = z_n$,

$$(-1)^n (\Delta(z_n) - \Delta(m_n)) = \zeta_n^2 F_n(q)/2,$$

$$F_n(q) \equiv (-1)^{n+1} \left[\ddot{\Delta}(z_n) + \zeta_n \int_0^1 (1-t)^2 \dddot{\Delta}(z_n + t\zeta_n)\, dt \right].$$

Using the properties of Δ, m_n, z_n we get that the function F_n is compact, real analytic and positive on H. From (2.22), (2.25), (2.34)–(2.35) and (3.2)–(3.3) we deduce that

$$F_n(q) = 1 + \ell^2(n), \qquad d_q F_n(q) = \ell^2(n), \qquad \text{as} \quad n \to \infty, \tag{4.12}$$

uniformly on $[0,1] \times B_{\mathbb{C}}(p, \varepsilon_p)$. Introduce the compact, real analytic and positive function $y_n = F_n(q)/f_n(q)$ on H. Then we can define a compact, real analytic and positive function $b_n = \sqrt{y_n}$ on H. Using (4.11)–(4.12) we obtain the asymptotic estimates (4.8).

The identities (1.5), (1.7) yield

$$\begin{aligned} \zeta_n^2 F_n/2 &= (-1)^n (\Delta(z_n) - \Delta(m_n)) \\ &= \cosh \sqrt{r_n} - \cosh \sqrt{r_{cn}} = (r_n - r_{cn})\, f_n(r_n, r_{cn})/2. \end{aligned}$$

Therefore, we obtain $r_n - r_{cn} = \zeta_n^2 y_n$, which implies (4.6), indeed

$$\begin{aligned} h_{sn}(q) &= |r_n - r_{cn}|^{1/2} \operatorname{sign}(z_n(q) - m_n(q)) \\ &= (z_n(q) - m_n(q))\, b_n(q), \qquad q \in H, \end{aligned}$$

therefore, the function $h_{sn}(u)$ is compact and real analytic on H. Identity (4.6) yields (4.7).

The identity (4.6) and the estimates (4.8), (2.34)–(2.35) and (3.2)–(3.3) imply (4.9)–(4.10). ∎

Now we need some modification of the well known Paley–Wiener result (see [Le]).

Lemma 4.3. *Let $f(z)$ be an even entire function of exponential type* 1 *and* $\int_{\mathbb{R}} |f(x)|^2\, dx < \infty$. *Assume $f(z_n) = 0$ for some sequence $\{z_n\}_1^\infty$ of positive distinct numbers such that $z_n = \pi n + \ell^p(n)$ for some $p \in [1, \infty)$ and $f(\pi n + \pi/2) = \ell^1(n)$. Then $f \equiv 0$.*

Proof. If f has an additional zero or the multiplicity of some z_n is greater than 1, then by the Paley–Wiener result (see [Le]), $f \equiv 0$. Assume such zero is absent, then the Hadamard factorization implies $f(z) = f(0) \prod (1 - (z/z_m)^2)$, where the product converges uniformly on bounded subsets of $\mathbb{C}$. A simple modification of Lemma 2 from [PT] (see p. 167) gives $f(\pi n + \pi/2) = f(0)\, n^{-1} (-1)^n (1 + o(1))$, $n \to \infty$ which yields $f(0) = 0$. ∎

We need the following result from analysis and we prove Theorem A. In fact, we repeat the corresponding proof of Theorem 5.1 from [KK1, KK2]. We give the detail proof since in [KK1] there was a mistake, corrected in [KK2].

THEOREM A. *Let H, H_1 be real separable Hilbert spaces with norms $\|\cdot\|$, $\|\cdot\|_1$. Suppose that a map $f\colon H\to H_1$ satisfies the following conditions:*

(i) *f is real analytic,*

(ii) *the operator $d_q f$ has an inverse for all $q\in H$,*

(iii) *there is a nondecreasing function $g\colon [0,\infty)\to[0,\infty)$, $g(0)=0$, such that $\|q\|\leqslant g(\|f(q)\|_1)$ for all $q\in H$,*

(iv) *there exists a basis $\{e_n\}_1^\infty$ of H_1 such that for all $n\geqslant 1$ the map $(f(\cdot),e_n)_1\colon H\to\mathbb{R}$ is compact*

(v) *for each $\xi>0$ the set $\{q\colon \sum n^p (f(q),e_n)_1^2<\xi\}$ is compact where $p>1$.*

Then f is a real analytic isomorphism between H and H_1.

Proof. Using Conditions (i), (ii) and the inverse function theorem, we see that the set $f(H)$ is open. We prove that it is also closed.

Suppose that $h_n=f(q_n)\to h$ strongly as $n\to\infty$. Then Condition (iii) yields $\|q_n\|\leqslant g(\|h_n\|_1)\leqslant g(\sup_{n>0}\|h_n\|_1)$. Hence there exists a subsequence $\{q_{n_m}\}_{m=1}^\infty$ such that $q_{n_m}\to q$ weakly as $m\to\infty$. Therefore, Condition (iv) implies $(f(q_{n_m}),e_k)_1\to(f(q),e_k)_1$ and $(f(q),e_k)_1=(h,e_k)_1$ for fixed $k\geqslant 1$. But $\{e_n\}_1^\infty$ is a basis of H_1, whence $h=f(q)$, and $f(H)=H_1$ since H_1 is connected.

We show that f is an injection. We introduce the sets

$$K_N=\{h\colon (h,e_n)_1=0 \text{ for } n>N\}\subset H_1$$

and $M_N=f^{-1}(K_N)\subset H$. The map f is a smooth local isomorphism on H, so that M_N is a real smooth submanifold of H (of dimension N).

We show that weak convergence in M_N implies strong convergence. Indeed, let $q_m\in M_N$ and $q_m\to q$ weakly as $m\to\infty$. Then $h_m=f(q_m)\in K_N$ and by Condition (iv), $h_m\to h$ strongly. Hence $\sum n^p(f(q_m),e_n)_1^2<C_0<\infty$ where the constant C_0 does not depend on m. Then by Condition (v), the sequence q_m is compact. Hence the topology on M_N, generated by the norm $\|\cdot\|$, coincides with the topology on M_N induced by the weak topology of H, that is, weak convergence implies strong convergence.

Denoting by E_N the set of points in K_N that have more than one pre-image, we see that E_N is open because f is a local isomorphism. But E_N is also closed. Indeed, suppose there are distinct points q_j and w_j in M_N such that $f(q_j)=f(w_j)\to h$ as $j\to\infty$. Then, using Condition (iii), we obtain two subsequences such that $q_{j_m}\to q$, $w_{j_m}\to w$, as $m\to\infty$, and Condition (iv) implies $q,\ w\in M_N$. If $q=w$, then $q_{j_m}=w_{j_m}$ for large m, because the map f is a local homeomorphism. Hence $q\neq w$ and E_N is closed. But $0\notin E_N$, whence $E_N=\varnothing$. Thus, $f\colon M_N\to K_N$ is an isomorphism.

Suppose that $f\colon H\to H_1$ is not an injection. Then some point $h\in H_1$ has at least two pre-images. Since f is a local homeomorphism, the same is true of every point in some neighbourhood of h and, consequently, for some $h^0\in K_N$ such that $(h^0, e_k)_1=(h, e_k)_1$ if $1\leqslant k\leqslant N$ and $(h^0, e_k)_1=0$ if $k>N$, where N is sufficiently large. But this contradicts the fact that $f\colon M_N\to K_N$ is an isomorphism. ∎

We are now ready to prove the our main theorem.

Proof of Theorem 1.1. We check all conditions of Theorem A for the mapping $h\colon H\to\ell^2\oplus\ell^2$.

(i) By Lemma 4.1–4.2 each function $h_n(\cdot)$, $n\geqslant 1$, is compact and real analytic on H, and the estimates (1.9)–(1.10) are fulfilled. The relations (1.9) show that the map $h(q)$ is locally bounded; by the uniform boundedness principle, $h(q)$ is real analytic.

(ii) Using the estimate (1.10), we see that $d_q h$ is the sum of the operator Φ and a compact operator for all $q\in H$. That is $d_q h-(1/\sqrt{2})\,\Phi$ is a compact operator; consequently, $d_q h$ is a Fredholm operator. We prove that the operator $d_q h$ is invertible by contradiction. Let $g\in H$ be a solution of the equation

$$(d_q h(q))\, g=0, \qquad \text{or} \quad \{(d_q h_n(q), g)=0, n\geqslant 1\}, \tag{4.13}$$

for some fixed $q\in H$. We introduce the even function

$$f(z)=\int_0^1 (\partial\Delta)(x, z, q)\ g(x)\,dx, \qquad z\in\mathbb{C}.$$

The function $r_n=h_{cn}^2+h_{sn}^2$ is analytic and (4.13) implies $(d_q r_n, g)=0$. Then (3.10) yields

$$f(z_n)=(-1)^n\,(d\cosh\sqrt{h_n}/dr_n)(d_q r_n, g)=0.$$

Hence $f(z_n)=0$ for any $n\geqslant 1$, and (3.2) implies $z_n=\pi n+\ell^d(n)$ as $n\to\infty$. Now we have to show that $f\in L^2(\mathbb{R})$. The asymptotic estimate (2.24) yields

$$f(z)=\int_0^1 (\partial\Delta(z,q))(t)\, g(t)\, dt$$
$$=\int_0^1\int_0^1 \cos z(2y-1)\, q(y+t)\, g(t)\, dt\, dy+\ell^1(n)$$
$$=\int_0^1 \cos z(2y-1)\, G(y)\, dy+\ell^1(n),$$
$$|z-\pi n|\leqslant\pi,\quad |n|\to\infty,$$

where $G(y)=\int_0^1 q(y+t)\, g(t)\, dt$. Hence $f\in L^2(\mathbb{R})$ and $f(\pi n+\pi/2)=\ell^1(n)$.

So, using (2.5) we see that all conditions of Lemma 4.3 are fulfilled, and we get $f\equiv 0$.

For fixed $q\in H$ we have 3 cases. First, let $h_{sn}=0$. Then by (4.7), $(d_q h_{sn}, g)=b_n(d_q z_n-d_q m_n, g)=0$ and by (3.1) we deduce that $(d_q z_n, g)=0$ and therefore $(d_q m_n, g)=0$.

Second, if $h_{cn}\neq 0$, $h_{sn}\neq 0$, then we have by relation (4.2)

$$(-1)^n \sinh h_{cn}(d_q h_{cn}, g)=\dot\Delta(m_n(q),q)(d_q m_n, g),$$

and then we have $(d_q m_n, g)=0$, since $f\equiv 0$ and $\dot\Delta(m_n)\neq 0$.

In the last case, let $h_{cn}=0$, $h_{sn}\neq 0$, then by (1.7), $\vartheta(1,m_n(q),q)=\varphi'(1,m_n(q),q)=(-1)^n$. Therefore, using the relations (2.11) and (2.12), we get

$$f(m_n(q))=(\partial\Delta(m_n(q),q),g)=A_n(d_q m_n(q),g),$$
$$A_n=-\theta'(1,z,q)\,\varphi'(1,z,q)\,\dot\varphi(1,z,q)\neq 0,\qquad z=m_n(q),$$

since $f=0$ and $\int_0^1 g(x)\, dx=0$. Then we have $(d_q m_n(q), g)=0$.

Hence we obtain $(d_q m_n(q), g)=0$, $(d_q h_{cn}, g)=0$, $n\geqslant 1$. Note that for any fixed $q\in H$ the vectors $\{(d_q h_{cn}), (d_q m_n), n\geqslant 1\}$ form a basis of H (see [CM]). Then $g=0$ and the operator $d_q h$ is invertible.

(iii) The estimates (1.8) were proved in [K1].

(iv) By Lemma 4.1–4.2, each mapping $h_n\colon H\to\mathbb{R}^2$, $n\geqslant 1$, is compact.

(v) We need some results about the Hill operator $T_1=-(d^2/dx^2)+V(x)$ in $L^2(\mathbb{R})$ with a real 1-periodic potential $V\in L^2(0,1)$. In the paper [K6] the following estimate has been proved

$$\|V\|\leqslant 2(1+\|h\|_{(1)})\,\|h\|_{(1)},\qquad \|h\|_{(1)}^2=\sum_{n\geqslant 1}(2\pi n)^2\,|h_n|^2.$$

Then the identity (1.17) yields $\|q'\|\leqslant 2(1+\|h\|_{(1)})\,\|h\|_{(1)}$. Hence, for each $\xi>0$ the set $\{q\colon \|h\|_{(1)}<\xi\}$ is compact.

Therefore, all conditions of Theorem A are fulfilled and then h is the real analytic isomorphism between H and $\ell^2\oplus\ell^2$. ∎

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